MMDA SEMINAR 3

1. Recap

Make sure you can shortly answer the following questions.

1. What do we call a simple function ?
2. What is the basic approximation property of positive simple functions with respect to positive measurable functions ?
3. Define the integral of a positive simple function.
4. Define the integral of a positive measurable function.
5. What do we call an integrable function ?
6. Define the integral of an integrable function.

2. Integral of a sum

Let (f_n) be a sequence of measurable and positive functions $f_n : (S, \mathcal{S}) \to (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ and let μ be a positive measure on $(S, \mathcal{S})$. Then, show that we always have

$$\int_S \sum_{n\geq 1} f_n \,\mathrm{d}\mu = \sum_{n\geq 1} \int_S f_n \,\mathrm{d}\mu.$$

3. Measures with densities

Consider three positive (and σ-finite) measures ν, μ, m on $(S, \mathcal{S})$. Suppose that $\nu \ll \mu$ and that $\mu \ll m$.

1. Show that $\nu \ll m$.
2. Show that

$$\frac{\mathrm{d}\nu}{\mathrm{d}m} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \cdot \frac{\mathrm{d}\mu}{\mathrm{d}m}$$

3. Let $A \in \mathcal{B}(\mathbb{R}^d)$ be such that $\lambda_d(A) \in (0, +\infty)$ and let $\mathcal{U}_A$ be the uniform measure on A.

3.a. Show that $\mathcal{U}_A \ll \lambda_d$.

3.b. Give the expression of

$$\frac{\mathrm{d}\mathcal{U}_A}{\mathrm{d}\lambda_d}$$

4. In this question we illustrate the importance of the σ-finite assumption. Show that the counting measure η on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is not σ-finite. Show that $\lambda_d \ll \eta$ but that λ_d does not have a density with respect to η: there exists no measurable and positive function $f : \mathbb{R}^d \to \mathbb{R}_+$ such that

$$\lambda_d(A) = \int_A f \,\mathrm{d}\eta.$$